Aesthetics and Pragmatism in the Proportions of the Yumi

John C. Neu

July 7, 2019

An enigmatic little note from a high level Kyudo sensei was forwarded to me after a recent seminar. It is a brief review of the Golden Ratio. An attached memo says that it "refers to the shape of the yumi."

In the fifth century BC, the Greek sculptor and mathematician Phidias proposed the division of a line segment into two with "the most beautiful proportions." Let's start by walking a path from feeling and intuition to that definite number, the Golden Ratio. Then let's see what it has to do with Kyudo. Let a and b denote lengths of the two segments, with a the greater, $a > b$. The division of a line into two defines not two, but *three* lengths, with the descending order,

$$\begin{aligned} a+b \qquad & (\textit{the total length}) \\ a \qquad & (\textit{the greater}) \\ b \qquad & (\textit{the lesser}) \end{aligned} \tag{1}$$

Comparison is often done by pairs, of which there are three:

$$\begin{aligned} a+b \quad & \textit{and} \quad b \\ a \quad & \textit{and} \quad b \\ a+b \quad & \textit{and} \quad b \end{aligned} \tag{2}$$

The first two comparisons are the "closest" to each other, comparing one length to the next smaller. The intuitive sense of "most beautiful proportions" might arise like this: "We have two perspectives, of comparing the total to the greater, and the greater to the lesser. Let's impose *symmetry* as a kind of equivalence between the two perspectives: The total stands in relation to the greater, the same as the greater to the lesser." The *numerical* comparison of two lengths is their ratio, so the quantitative expresssion is

$$\frac{a+b}{a} = \frac{a}{b}. \tag{3}$$

Defining x as the ratio $\frac{a}{b}$, this equation is equivalent to

$$1 + \frac{1}{x} = x, \tag{4}$$

whose *positive* solution is the Golden Ratio

$$x = \phi := \frac{1+\sqrt{5}}{2} \approx 1.61803... \tag{5}$$

So, we "know" the Golden Ratio. Does knowing alone clarify its persistence in artistic and architectural works over the two and one half millenia from Phidias to the present? For instance, the Golden Ratio informs the proportions of work spaces on computer screens. My belief is that symmetries, in particular symmetries underlying geometric proportions, are deeply moving to the human heart. Why is that? In a broader sense, *symmetries refer to aspects of phensomena that remain the same under a change of perspective.* Whispered intimations of connection and unity underlying apparent seperatness and diversity.

Now, Kyudo! The first thing that comes to (my) mind is the *proportion between yumi lengths above and below the top of the grip.* Let's denote these lengths a and b respectively. The first two columns of table 1 record measurements of a and b of five Yonsun yumi by a respected yumishi . The third and fourth colums are computed values of proportions $\frac{a+b}{a}$ and $\frac{a}{b}$.

I don't know that yumishi intentionally impose grip placement according to the Golden Ratio. Nevertheless, it is very nearly realized for the five yumi. Phidias originally applied the Golden Ratio to set proportions in human figure sculptures of the Parthenon temple. There is a tradition of recognizing (or seeking!) the Golden Ratio in idealized human proportions, such as in Leonardo da Vinci's "Vitruvian Man." On page 132 of the Kyudo Kyohan, there is an idealized line drawing of Kai. Superimposed upon it, there is the central vertical axis of the Kyudoka, and the three horizontal lines of shoulders, hips and feet, as in the Sanjumonji (three crosses). In the

Table 1: Proportions of lengths above and below grip

$a\ (cm)$	$b\ (cm)$	$\frac{a+b}{a}$	$\frac{a}{b}$
144.0	88.7	1.616	1.623
144.8	89.5	1.618	1.618
145.4	89.0	1.612	1.634
145.3	88.9	1.612	1.634
142.7	90.4	1.633	1.578

figure, the elevation of the shoulder line above the feet is 76.5 mm, and the elevation of the hip line, 47.5 mm. The ratio of elevations is 1.611. Is this intentional? Does it it arise "naturally" from an unspoken consensus over generations and centuries?

There is the temptation to seek further realizations of the Golden Ratio in other proportions hidden in yumi geometry, or postures of the Kyudoka. Instead, I propose to examine node placements and proportions of the five curves by joining intuition and reason, somewhat in the spirit of Phidias, but different: Pragmatic issues of dealing with the materials (bamboo!) impose themselves.

Let's begin by reviewing some features of yumi design. Apparently, they were *chosen* long ago and maintained by tradition over the centuries. I'll refer to them as *accepted design precepts*. They are very reliably present in all the five yumi's I've measured for this discourse. The design precepts are summarized in figure 1. To the right in figure 1, there is a photograph of the third of yonsun yumi in its unbraced configuration. To the left, a schematic drawing marking node positions and contact points between curves.

First, the nodes: There are six exposed belly nodes marked by circles, and seven exposed back nodes marked by squares. Certain observations turn out to be crucial: (i) The top of the grip is located at the fourth belly node from the top. (ii) Each belly node is close to the halfway point between adjacent back nodes. (iii) The upper tip of yumi is just short of where the next back node would be. (iv) The lower tip of the yumi extends just beyond where the next belly node would be. On some of the five yumi, the belly bamboo continues under the lower strike plate, and the tell-tale of the "buried" belly node is a disturbance in the bamboo grain when you look from the side.

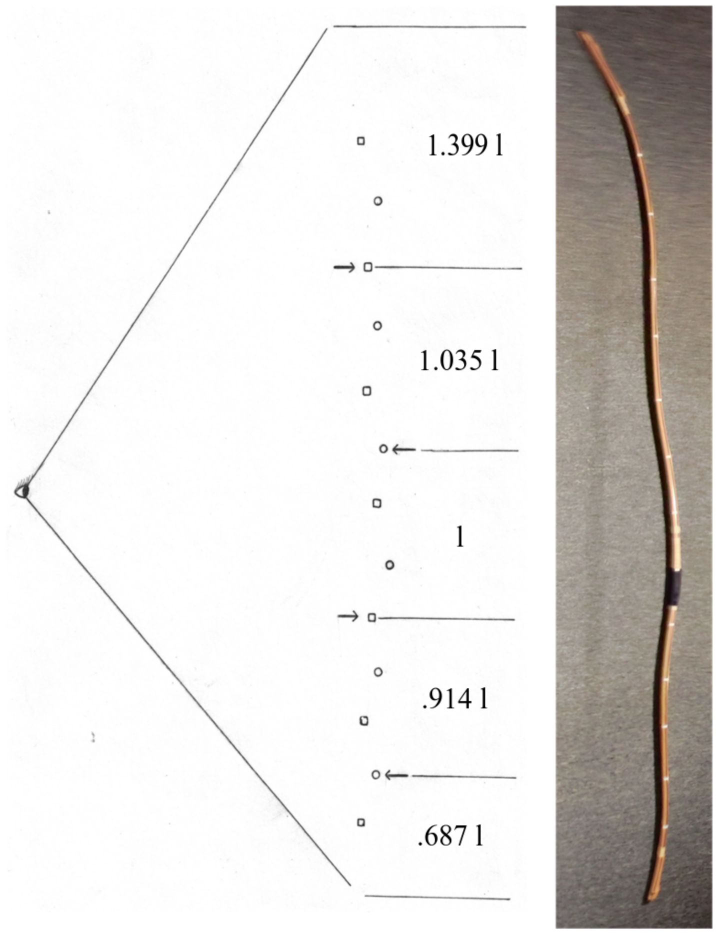

Figure 1

There is typically another "buried" belly node underneath the upper strike plate.

Next, the curves: Imagine facing the yumi so you gaze along its back. From this perspective, the top and bottom curves are concave. This appears to be a "natural" choice consistent with most archery traditions everywhere and throughout the ages. The curve containing the grip is concave as well, as if yumishi wanted to impart some extra Ikasu ("life") about the grip. The *mathematics* of curves dictates that concave and convex curves must alternate, so there must be at least one convex curve above the grip, and at least one other below. This makes five curves in all.

In mathematics, the contact between a concave curve and an adjacent convex curve is called an *inflection point*. The placing of inflection points is subtle and delicate: Small modifications of the yumi shape can drastically alter their positions. How in practice are the contact points between adjacent curves determined? Further inspection of the five yumi suggests that these contact points are close to alternating back and belly nodes, marked by arrows in figure 1. By inspection of figure 1, we surmize the number of internode intervals alloted to each curve, starting from the top. These are recorded in table 2.

Table 2: Internode spacings in each curve

Curve	Internode spacings
1	(*slightly less than*) 2
2	3/2
3	3/2
4	3/2
5	(*slightly more than*) 1

The observations (i) -(iv) of node positions and contact points between curves stongly constrain yumi proportions. As a first approximation, assume that the spacings between bamboo nodes are uniform. We actually know better: Node spacings *increase* as we ascend the stalk, and we'll return to this. For now, let's see the approximate picture that emerges: From the observations (i)-(iv), we deduce that there are (slightly less than) four and one half internodal spacings from the top of the grip to the upper tip, and (slightly more than) three below. With uniform node spacing, the approximation to the proportion between the lengths above and below the grip is

$$\frac{a}{b} \approx 1.5, \tag{6}$$

which is about 7.4% below the measured values in table 1. Assuming uniform node spacing, the numbers in table 2 are the lengths of curves one through four relative to the bottom curve. Alternatively, we observe that the *sum* of values in table 2 (which is 7.5) expreses the length of yumi in units of node spacing. Dividing each entry of table 2 by 7.5 approximates the curve lengths as *fractions* of the total yumi length. Table 3 presents these values, along with measured values for each of the five yumi. The last column tabulates

Table 3: Fractional lengths of the curves

Curves	Simple theory	yumi 1	yumi 2	yumi 3	yumi 4	yumi 5	geometric mean
1	.267	.277	.279	.282	.266	.283	.277
2	.200	,211	.200	.204	..211	.202	.205
3	.200	.197	.198	.194	..201	.198	.198
4	.200	.186	.182	.174	..193	.173	.181
5	.134	.134	.136	.139	..123	.148	.136

the *geometric means* of the measured fractional lengths for each curve. For instance, the geometric mean of the measured fractional lengths of curve one for the five yumi is

$$\{(.277)(.279)(.282)(.266)(.283)\}^{\frac{1}{5}} \approx .277. \tag{7}$$

The geometric mean (instead of the usual arithmetic mean) is most appropriate for *comparing proportions*. This is because the geometric mean of ratios is the ratio of geometric means. (The arithmetic mean of ratios is generally not equal to the ratio of arithmetic means.) For instance, the geometric mean of the proportion between curves one and two is $.227/.205 \approx 1.35$, close to the "naive" value of .134 based on uniform node spacing.

The closeness of the "naive" curve proportions to what is actually measured speaks for the *robustness of the design precepts*: Materials and yumishi vary, as they must. Nevertheless proportions come out nearly the same time and time again.

Let's now concentrate on the *deviations* from the oversimplified proportions, as set forth in the "simple theory" column of table 3. Each of the "middle" curves two, three and four span close to 3/2 internodal spacings, so the progressive decrease of observed fractional lengths .205, .198, .181 is a symptom of the internodal spacings decreasing as we descend. The departures of measured proportions from the naive proportions informs the departure of the proportion of yumi lengths between yumi lengths above and below the grip from the naive value of 1.5 in (6). Let l be the length of curve three (the curve containing the grip). Then the lengths of the remaining curves according to the geometric averaged fractional lengths in the last column of table 3 are

$$\begin{aligned} l_1 &\approx \frac{.277}{.198}\, l \approx 1.399\, l, \\ l_2 &\approx \frac{.205}{.198}\, l \approx 1.035\, l, \\ l_4 &\approx \frac{.181}{.198}\, l \approx .914\, l, \\ l_5 &\approx \frac{.136}{.198}\, l \approx .687\, l. \end{aligned} \tag{8}$$

These curve lengths are labeled in figure 1 for easy visual reference. We estimate the lengths a and b of yumi above and below the grip in terms of these:

$$\begin{aligned} a &\approx l_1 + l_2 + \frac{2}{3}\, l_3 \approx (1.399 + 1.035 + .667)\, l \approx 3.101\, l, \\ b &\approx \frac{1}{3}\, l_3 + l_4 + l_5 \approx (.334 + .914 + .687)\, l \approx 1.935\, l. \end{aligned} \tag{9}$$

From these, we compute

$$\frac{a+b}{a} \approx 1.603, \ \frac{a}{b} \approx 1.603. \tag{10}$$

To summarize: A sense of aethetic mystiscm is implicit in many discussions of the Golden Ratio and its actual occurances in real life. The exercise just completed here raises a possibility: The closeness of the yumi proportions $\frac{a+b}{a}$ and $\frac{a}{b}$ to the Golden Ratio emerge *naturally* from the (closely followed) design precepts and the increase of internode spacing as you ascend the bamboo stalk.